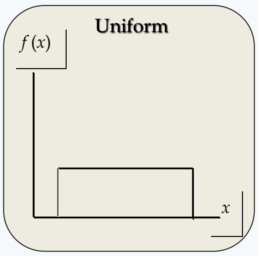
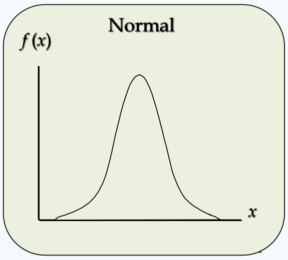
# Chapter 8: Continuous Probability Distributions

## Chapter 8: Continuous Probability Distributions




### Introduction

This chapter completes our presentation of probability by introducing continuous random variables and their distributions, which are used to calculate the probability associated with an **interval variable**.

We introduce probability density functions and use the uniform density function to demonstrate how probability is calculated.

Later, we focus on the normal distribution, one of the most important distributions because of its role in the development of statistical inference.

2

### Continuous Random Variable

A **continuous random variable** can assume an uncountable number of values, and differs from a discrete variable in several respects:

- We cannot list the possible values of a continuous random variable because there is an infinite number of them.
- Because there is an infinite number of values, the probability of each individual value is virtually 0.
- Thus, we can only calculate the probability of a range of values.

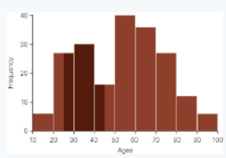
The following requirements apply to a **probability density function** $f(x)$ whose range is $a \leq x \leq b$.

1. $f(x) \geq 0$ for all $x$ between $a$ and $b$.
2. The total area under the curve between $a$ and $b$ is 1.

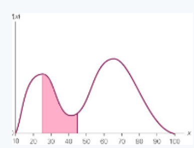
3

### Probability Density Functions

Estimated Probability between 25 and 45 Years from the Histogram in Example 3.1




**Probability Density Function (PDF)** for Example 3.1



- In Example 3.1, we could only estimate the relative frequency that an ACBL member is between 25 and 45 by multiplying base × height for each dark brown rectangle, knowing that the sample size is 200.
- If we can determine a probability density function, which is essentially a smooth histogram drawn from a large population and made of infinite small rectangles, we can calculate the exact probability.

4

## Uniform Distribution

To illustrate how we find the area under the curve that describes a probability density function, consider the **uniform probability distribution**, also called the **rectangular probability distribution**, which is described by the function:

$$f(x) = \frac{1}{b-a} \text{ where } a \le x \le b$$

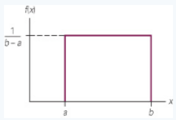
To calculate the probability that $X$ falls between $x_1$ and $x_2$, simply determine the area in the rectangle whose base is $x_2 - x_1$ and whose height is $1/(b-a)$.

Thus:

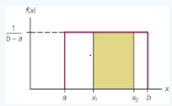
$$P(x_1 \le X \le x_2) = Base \times Height$$

$$= (x_2 - x_1)\frac{1}{b-a}$$

Uniform Distribution



$P(x_1 \le X \le x_2)$



5

## Uniform Probability Distribution

- Expected Value of $x$

$$\boxed{E(x) = (a + b)/2}$$

- Variance of $x$

$$\boxed{Var(x) = (b - a)^2/12}$$

where: $a$ = smallest value the variable can assume

$b$ = largest value the variable can assume

6

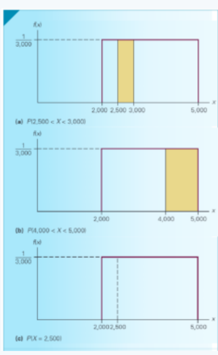
## Uniformly Distributed Gasoline

The amount of gasoline sold daily at a service station is uniformly distributed between 2,000 and 5,000 gallons. Find the following:

a. Probability that daily sales fall between 2,500 and 3,000 gallons.

b. Probability that the service station sells at least 4,000 gallons.

c. Probability that the station sells exactly 2,500 gallons.



7

## Uniformly Distributed Gasoline Sales

**Solution**

The probability density function is:

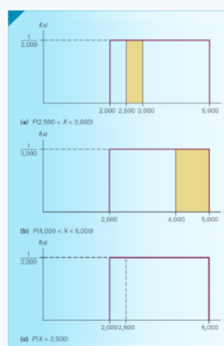
$$f(x) = \frac{1}{5{,}000 - 2{,}000}$$

$$= \frac{1}{3{,}000} \text{ when } 2{,}000 \le x \le 5{,}000$$

a. $P(2{,}500 \le x \le 3{,}000) = (3{,}000 - 2{,}500)\frac{1}{3000} = .1667$

b. $P(x \ge 4{,}000) = (5{,}000 - 4{,}000)\frac{1}{3000} = .3333$

c. $P(x = 2{,}500) = 0$



**Note:** $P(2{,}500 \le x \le 3{,}000) = P(2{,}500 < x < 3{,}000)$

8

## Uniform Probability Distribution Example

Slater buffet customers are charged for the amount of salad they take. Sampling suggests that the amount of salad taken is uniformly distributed between 5 ounces and 15 ounces.

- Uniform Probability Density Function

$$f(x) = 1/10 \quad \text{for } 5 \leq x \leq 15$$
$$= 0 \quad \text{elsewhere}$$

where:

$x$ = salad plate filling weight

9

## Uniform Probability Distribution

- Expected Value of $x$

$$E(x) = (a + b)/2$$
$$= (5 + 15)/2$$
$$= 10$$

- Variance of $x$

$$Var(x) = (b - a)^2/12$$
$$= (15 – 5)^2/12$$
$$= 8.33$$

10

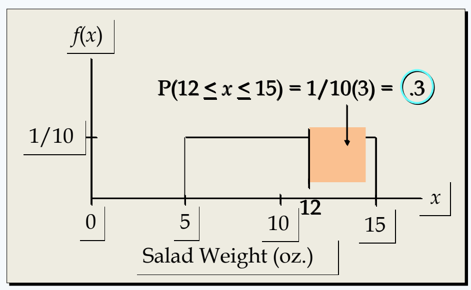
## Uniform Probability Distribution

- Uniform Probability Distribution for Salad Plate Filling Weight


11

## Uniform Probability Distribution

What is the probability that a customer will take between 12 and 15 ounces of salad?




12

## Uniform Probability Distribution
### (Another Example)

**Example:** Flight time of an airplane traveling from Chicago to New York

Suppose the flight time can be any value in the interval from 120 minutes to 140 minutes.

Uniform Probability Density Function

$$f(x) = 1/20 \quad \text{for } 120 \leq x \leq 140$$
$$= 0 \quad \text{elsewhere}$$

where:

$x$ = Flight time of an airplane traveling from Chicago to New York

13

## Uniform Probability Distribution
### Another Example

Expected Value of $x$

$$E(x) = (a + b)/2$$
$$= (120 + 140)/2$$
$$= 130$$

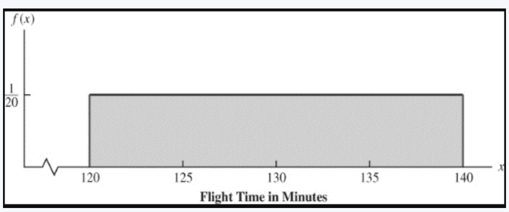
Variance of $x$

$$Var(x) = (b - a)^2/12$$
$$= (140 – 120)^2/12$$
$$= 33.33$$

14

## Uniform Probability Distribution

**Example:** Flight time of an airplane traveling from Chicago to New York



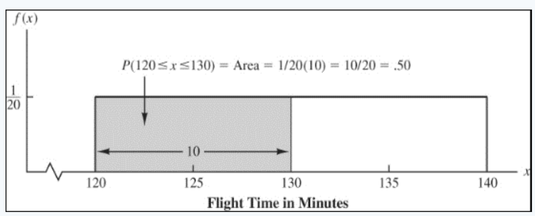

15

## Uniform Probability Distribution

**Example:** Flight time of an airplane traveling from Chicago to New York

Probability of a flight time between 120 and 130 minutes

$$P(120 \leq x \leq 130) = 1/20(10) = .5$$




16

## Area as a Measure of Probability

- The area under the graph of $f(x)$ and probability are identical.
- This is valid for all continuous random variables.
- The probability that $x$ takes on a value between some lower value $x_1$ and some higher value $x_2$ can be found by computing the area under the graph of $f(x)$ over the interval from $x_1$ to $x_2$.

17

## Normal Distribution

The normal distribution is the most important of all probability distributions because of its crucial role in statistical inference.

**Normal Density Function**

The probability density function of a **normal random variable** is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

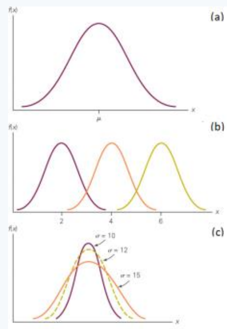
where $e$ = 2.71828... and $\pi$ = 3.14159...

The normal distribution is described by the mean $\mu$ and the standard deviation $\sigma$.

a. Notice how a normal distribution is symmetric about its mean and the random variable ranges between $-\infty$ and $+\infty$.

b. Changing the value of $\mu$ shifts the distribution left or right.

c. Increasing the value of $\sigma$ makes the distribution wider.

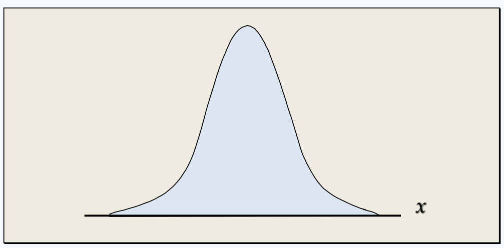



18

## Normal Probability Distribution

- Characteristics

The distribution is <u>symmetric</u>; its skewness measure is zero.

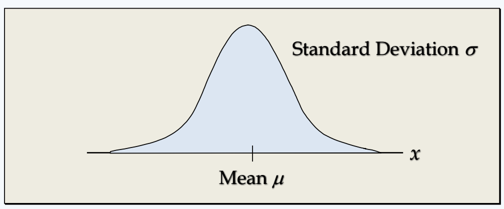



19

## Normal Probability Distribution

- Characteristics

The entire family of normal probability distributions is defined by its <u>mean</u> $\mu$ and its <u>standard deviation</u> $\sigma$.

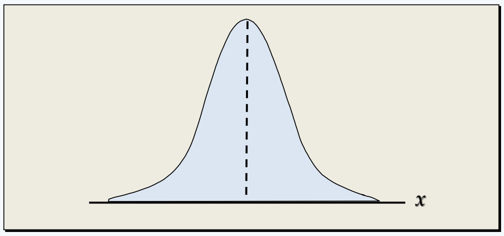



20

## Normal Probability Distribution

- Characteristics

The highest point on the normal curve is at the mean, which is also the median and mode.

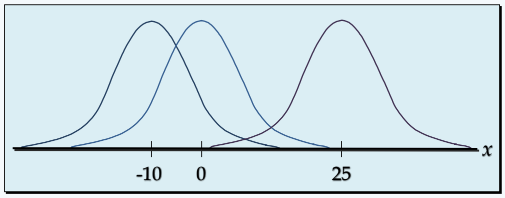



21

## Normal Probability Distribution

- Characteristics

The mean can be any numerical value: negative, zero, or positive.

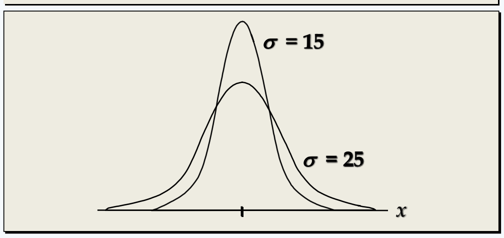



22

## Normal Probability Distribution

- Characteristics

The standard deviation determines the width of the curve: larger values result in wider, flatter curves.

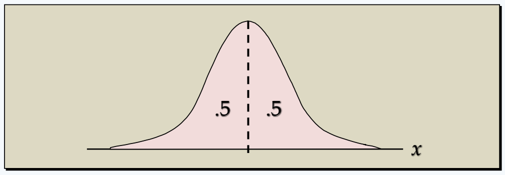



23

## Normal Probability Distribution

- Characteristics

Probabilities for the normal random variable are given by areas under the curve. The total area under the curve is 1 (.5 to the left of the mean and .5 to the right).




24

## Normal Probability Distribution

- Characteristics (basis for the empirical rule)

68.26% of values of a normal random variable are within +/- 1 standard deviation of its mean.

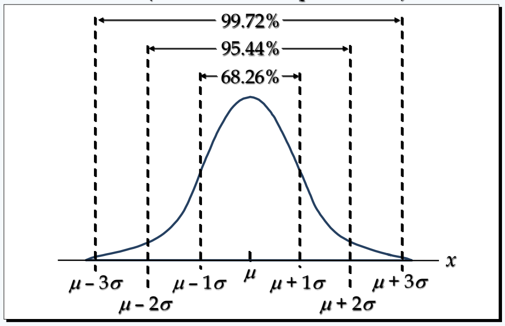
95.44% of values of a normal random variable are within +/- 2 standard deviations of its mean.

99.72% of values of a normal random variable are within +/- 3 standard deviations of its mean.

25

## Normal Probability Distribution

- Characteristics (basis for the empirical rule)

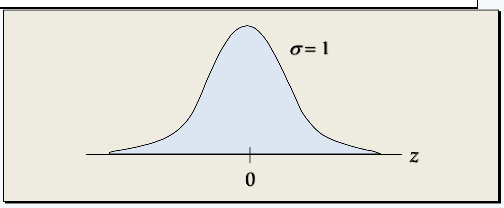



26

## Standard Normal Probability Distribution

- Characteristics: A random variable having a normal distribution with a mean of 0 and a standard deviation of 1 is said to have a standard normal probability distribution.

The letter $z$ is used to designate the standard normal random variable.




27

## Standard Normal Probability Distribution

- Converting to the Standard Normal Distribution

$$z = \frac{x - \mu}{\sigma}$$

We can think of $z$ as a measure of the number of standard deviations $x$ is from $\mu$.

28

## Example – Calculating Normal Probabilities

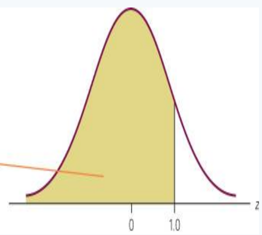
To calculate probabilities of the type *P(Z < z)* we can use the standard normal probability table (available on Canvas or Table 3 in Appendix B of the textbook). The table lists the cumulative probabilities for values of $z$ ranging from −3.09 to +3.09.

**Standard Normal Cumulative Probability Table**

We find the desired probability in the table's left margin:

$P(X < 1{,}100) = P(Z < 1.00) = .8413$

| z | .00 | .01 | .02 |
|---|---|---|---|
| 0.8 | .7881 | .7910 | .7939 |
| 0.9 | .8159 | .8186 | .8212 |
| 1.0 | .8413 | .8438 | .8461 |
| 1.1 | .8643 | .8665 | .8686 |
| 1.2 | .8849 | .8869 | .8888 |




29

## Example – Standardize the Normal Variable

Suppose that the daily demand for regular gasoline at a gas station is normally distributed with a mean of 1,000 gallons and a standard deviation of 100 gallons.

The station manager has just opened for business and noted that there is exactly 1,100 gallons of regular gasoline in storage.

The manager would like to know the probability that there will be enough regular gasoline to satisfy today's demands before the new supply.

**Solution**

We label the demand for regular gasoline as $X$, and we want to find
*P(X < 1,100)*, as shown to the top right.

First, we standardize $X$:

$$P(X < 1{,}100) = P\left(\frac{X-\mu}{\sigma} < \frac{1{,}100 - 1{,}000}{100}\right) = P(Z < 1.00)$$

As shown in the distribution.

*P(X < 1,100)*

*P(Z < 1.00)*

30

## Calculating Probabilities

To determine the probability that the standard normal random variable is greater than some value of $z$, we can use the complement rule.

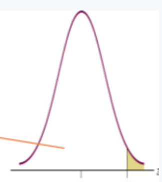
For example, to find the probability that $Z$ is greater than 1.80, we write:

$P(Z > 1.80) = 1 - P(Z \leq 1.80) = 1 - P(Z < 1.80)$

Again, in the table's left margin we find:

$P(Z > 1.80) = 1 - .9641 = .0359$

| z | .00 | .01 | .02 |
|---|---|---|---|
| 1.6 | .9452 | .9463 | .9474 |
| 1.7 | .9554 | .9564 | .9573 |
| 1.8 | .9641 | .9649 | .9656 |
| 1.9 | .9713 | .9719 | .9726 |
| 2.0 | .9772 | .9778 | .9783 |




31

## Probability of an Interval

To determine the probability that a standard normal random variable lies between two values of z.

For example, we can obtain the probability of the interval between −0.71 and 0.92 by calculating the difference between two cumulative probabilities:

$P(-0.71 < Z < 0.92)$
$= P(Z < 0.92) - P(Z < -0.71)$

From the table to the right, we find:

$P(Z < -0.71) = .2389$

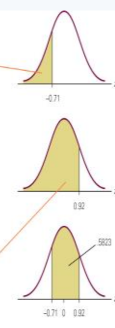
And

$P(Z < 0.92) = .8212$

Thus:

$P(-0.71 < Z < 0.92) = .8212 - .2389$
$= .5823$

| z | .00 | .01 | .02 |
|---|---|---|---|
| −0.8 | .2119 | .2090 | .2061 |
| −0.7 | .2420 | .2389 | .2358 |
| −0.6 | .2743 | .2709 | .2676 |
| −0.5 | .3085 | .3050 | .3015 |
| −0.4 | .3446 | .3409 | .3372 |
| −0.3 | .3821 | .3783 | .3745 |
| −0.2 | .4207 | .4168 | .4129 |
| −0.1 | .4602 | .4562 | .4522 |
| −0.0 | .5000 | .4960 | .4920 |
| 0.0 | .5000 | .5040 | .5080 |
| 0.1 | .5398 | .5438 | .5478 |
| 0.2 | .5793 | .5832 | .5871 |
| 0.3 | .6179 | .6217 | .6255 |
| 0.4 | .6554 | .6591 | .6628 |
| 0.5 | .6915 | .6950 | .6985 |
| 0.6 | .7257 | .7291 | .7324 |
| 0.7 | .7580 | .7611 | .7642 |
| 0.8 | .7881 | .7910 | .7939 |
| 0.9 | .8159 | .8186 | .8212 |
| 1.0 | .8413 | .8438 | .8461 |




32

## Finding Values of Z

There is a family of problems that requires us to determine the value of $Z$ given a probability.

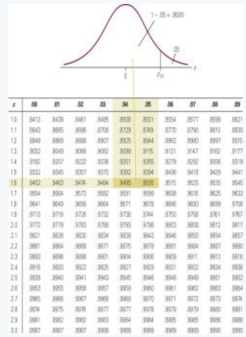
We use the notation $Z_A$ to represent the value of $z$ such that the area to its right under the standard normal curve is $A$; that is:

$P(Z > Z_A) = A$


33

## Finding Z for .05

Find the value of a standard normal random variable such that the probability that the random variable is greater than this quantity is 5% ($Z_{.05}$). $P(Z > Z_{.05}) = 0.05$



**Solution**

The probability that $z$ is less than $Z_{.05}$ must be
$1 - .05 = .9500$. To find $Z_{.05}$, we locate .9500 in the table.

As you can se from the figure to the right, there are two values of $Z$ that are equally close: .9495 and .9505.

Those two probabilities correspond to the $Z$-values of 1.64 and 1.65, respectively.

Thus, we can say that: $Z_{.05} = 1.645$.

34

## Finding the value of x

Question: Knowing that the mean of this distribution is 8 and the standard deviation is 1.6. What is the value of the variable corresponding to this standard normal random variable.

In other words, we are asked to find the X corresponding to this specific value of Z:

$Z=(X-\mu)/\sigma \rightarrow X = \sigma Z + \mu = 1.6 * 1.645 + 8 = 2.632 + 8 = 10.632$

We say that we unstandardized Z.

35

## Z-value and percentiles

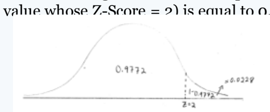
Example, for z = 2:

- According to our Z-Table, the area to the left of Z=2 is equal to 0.9772
- This means that the area to the right of Z=2 must be equal to 1 – 0.9772 = 0.0228
- This means that:

(i) The value in our distribution whose Z-Score = 2 is greater than 97.72% of the values in the distribution, and less than 2.28% of the values in the distribution.

(ii)The value (x) whose Z-Score = 2 is **97.72nd percentile**

(iii)The probability that a randomly selected value from our distribution has a Z-Score of less than 2 (in other words, the probability it is smaller than the value whose Z-Score = 2) is equal to 0.9772, and the probability that a random value has a Z-score of greater than 2 (the probability that that it is greater than the value whose Z-Score = 2) is equal to 0.0228




36

## Using excel to compute standard normal probabilities

Excel has two functions for computing probabilities and *z* values for a standard normal probability distribution.

- NORM.S.DIST function computes the cumulative probability given a *z* value.
- NORM.S.INV function computes the *z* value given a cumulative probability.

"S" in the function names reminds us that these functions relate to the standard normal probability distribution.

37

## Using Excel to compute Standard Normal Probabilities

❑Excel Formula Worksheet

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | Probabilities: Standard Normal Distribution | | |
| 3 | | $P(z <= 1)$ | =NORM.S.DIST(1,TRUE) | |
| 4 | | $P(-.5 <= z <= 1.25)$ | =NORM.S.DIST(1.25, TRUE) - NORM.S.DIST(-0.5, TRUE) | |
| 5 | | $P(-1 <= z <= 1)$ | =NORM.S.DIST(1, TRUE) - NORM.S.DIST(-1, TRUE) | |
| 6 | | $P(z >= 1.58)$ | =1 - NORM.S.DIST(1.58,TRUE) | |

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | Probabilities: Standard Normal Distribution | | | |
| 3 | | $P(z <= 1)$ | 0.8413 | | |
| 4 | | $P(-.5 <= z <= 1.25)$ | 0.5858 | | |
| 5 | | $P(-1 <= z <= 1)$ | 0.6827 | | |
| 6 | | $P(z >= 1.58)$ | 0.0571 | | |

38

## Using Excel to compute Standard Normal Probabilities

❑Excel Formula Worksheet

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | Finding z-values given Probabilities | | |
| 3 | | z value with .10 in upper tail | =NORM.S.INV(0.9) | |
| 4 | | z value with .025 in upper tail | =NORM.S.INV(0.975) | |
| 5 | | z value with .025 in lower tail | =NORM.S.INV(0.025) | |

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | Finding z-values given Probabilities | | |
| 3 | | z value with .10 in upper tail | 1.28 | |
| 4 | | z value with .025 in upper tail | 1.96 | |
| 5 | | z value with .025 in lower tail | -1.96 | |

39

## Standard Normal Probability Distribution

**Example:** Grear Tire Company Problem

Grear Tire company has developed a new steel-belted radial tire to be sold through a chain of discount stores. But before finalizing the tire mileage guarantee policy, Grear's managers want probability information about the number of miles of tires will last.

It was estimated that the mean tire mileage is 36,500 miles with a standard deviation of 5000. The manager now wants to know the probability that the tire mileage *x* will exceed 40,000.

$P(x > 40,000) = ?$

40

## Standard Normal Probability Distribution

**Example:** Grear Tire Company Problem

Solving for the Probability

- Step 1: Convert *x* to standard normal distribution.

$$z = (x - \mu)/\sigma$$
$$= (40{,}000 – 36{,}500)/5{,}000$$
$$= .7$$

- Step 2: Find the area under the standard normal curve to the left of $z = .7$.

41

## Standard Normal Probability Distribution

**Example:** Grear Tire Company Problem

Cumulative Probability Table for the Standard Normal Distribution

| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| . | . | . | . | . | . | . | . | . | . | . |
| .5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| .6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| .7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| .8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| .9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| . | . | . | . | . | . | . | . | . | . | . |

$$P(z \leq .7) = .7580$$

42

## Standard Normal Probability Distribution

**Example:** Grear Tire Company Problem

Solving for the Probability

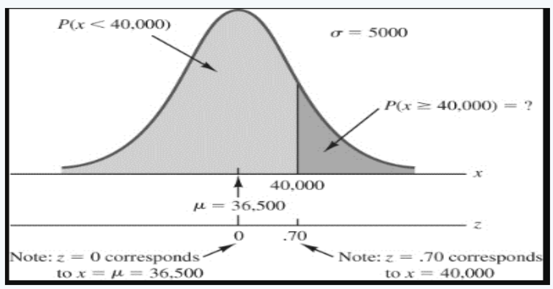
- Step 3: Compute the area under the standard normal curve to the right of $z = .7$

$$P(z > .7) = 1 – P(z \leq .7)$$
$$= 1- .7580$$
$$= .2420$$

43

## Standard Normal Probability Distribution

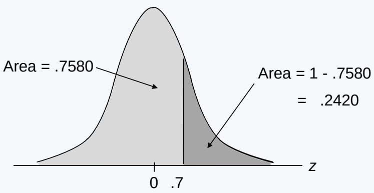
**Example:** Grear Tire Company Problem




44

## Standard Normal Probability Distribution

**Example:** Grear Tire Company Problem




45

## Standard Normal Probability Distribution

- Example: Pep Zone

Pep Zone sells auto parts and supplies including a popular multi-grade motor oil. When the stock of this oil drops to 20 gallons, a replenishment order is placed. The store manager is concerned that sales are being lost due to stockouts while waiting for a replenishment order.
It has been determined that demand during replenishment lead-time is normally distributed with a mean of 15 gallons and a standard deviation of 6 gallons.

*The manager would like to know the probability of a stockout during replenishment lead-time. In other words, what is the probability that demand during lead-time will exceed 20 gallons?*

$P(x > 20) = ?$

46

## Standard Normal Probability Distribution

- Solving for the Stockout Probability

Step 1: Convert $x$ to the standard normal distribution.

$$z = (x - \mu)/\sigma$$
$$= (20 - 15)/6$$
$$= .83$$

Step 2: Find the area under the standard normal curve to the left of $z = .83$.

47

## Standard Normal Probability Distribution

- Solving for the Stockout Probability

Step 3: Compute the area under the standard normal curve to the right of $z = .83$.

$$P(z > .83) = 1 - P(z \le .83)$$
$$= 1 - .7967$$
$$= .2033$$

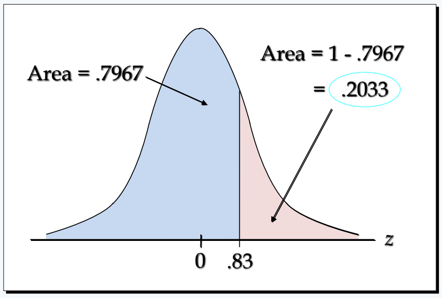
Probability of a stockout

$P(x > 20)$

48

## Standard Normal Probability Distribution

- **Solving for the Stockout Probability**

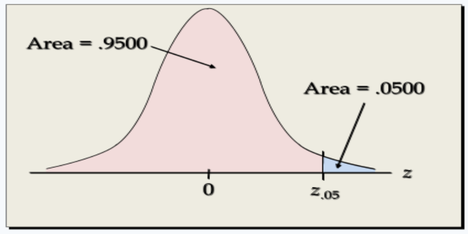



49

## Standard Normal Probability Distribution

- **Standard Normal Probability Distribution**

If the manager of Pep Zone wants the probability of a stockout during replenishment lead-time to be no more than .05, what should the reorder point be?




50

## Standard Normal Probability Distribution

- **Solving for the Reorder Point**

Step 2: Convert $z_{.05}$ to the corresponding value of $x$.

$$z = (x - \mu)/\sigma$$
$$z\sigma = (x - \mu)$$
$$x = \mu + z\sigma$$

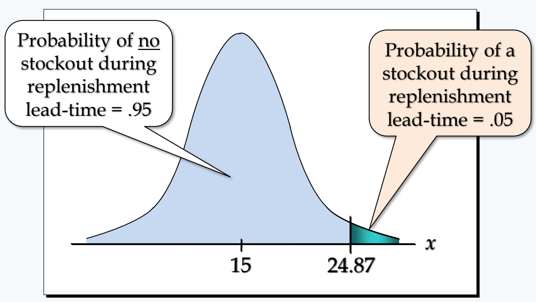
$$x = \mu + z_{.05}\sigma$$
$$= 15 + 1.645(6)$$
$$= 24.87 \text{ or } 25$$

A reorder point of 25 gallons will place the probability of a stockout during leadtime at (slightly less than) .05.

51

## Normal Probability Distribution

- **Solving for the Reorder Point**




52

## Standard Normal Probability Distribution

- **Solving for the Reorder Point**

By raising the reorder point from 20 gallons to 25 gallons on hand, the probability of a stockout decreases from about .20 to .05.

This is a significant decrease in the chance that Pep Zone will be out of stock and unable to meet a customer's desire to make a purchase.

53

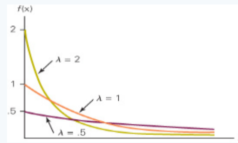
## Exponential Distribution

**Exponential Probability Density Function**

- A random variable $X$ is exponentially distributed if its probability density function is given by:
  - $f(x) = \lambda e^{-\lambda x}\ x \geq 0$
- where $e$ = 2.71828… and $\lambda$ is the parameter of the distribution. The image to the right show the exponential distribution for different values of $\lambda$.
- The mean of an exponential random variable is equal to its standard deviation:
  - $\mu = \sigma = 1/\lambda$

**Probability Associated with an Exponential Random Variable**

- If $X$ is an exponential random variable:
- $P(X > x) = e^{-\lambda x}$
- $P(X < x) = 1 - e^{-\lambda x}$
- $P(x_1 < X < x_2) = e^{-\lambda x_2} - e^{-\lambda x_1}$




54

## Example – Lifetimes of Alkaline Batteries

The lifetime of an alkaline battery (measured in hours) is exponentially distributed with $\lambda$ = .05.

a. What are the mean and standard deviation of the battery's lifetime?

b. Find the probability that a battery will last between 10 and 15 hours.

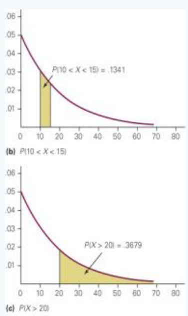
c. What is the probability that a battery will last for more than 20 hours?

**Solution**

a. $\mu = \sigma = 1/\lambda$ = 1/.05 = 20 hours

b. Let $X$ be the lifetime of a battery. Then:

$P(10 < X < 15) = e^{-.05(10)} - e^{-.05(15)} = e^{-0.5} - e^{-.75}$






55

## Example – Supermarket Checkout Counter

A checkout counter at a supermarket completes the process according to an exponential distribution with a service rate of 6 per hour.

A customer arrives at the checkout counter.

Find the probability of the following events.

a. The service is completed in fewer than 5 minutes.

b. The customer leaves the checkout counter more than 10 minutes after arriving.

c. The service is completed in a time between 5 and 8 minutes.

**Solution**

Because all the questions are about services completed in minutes, let us convert the service rate also in minutes:

$\lambda$ = 6 / hour = .1 / minute.

Thus:

a. $P(X < 5) = 1 - e^{-.1(5)} = 1 - e^{-0.5} = .3935$

b. $P(X > 10) = e^{-.1(10)} = e^{-1} = .3679$

c. $P(5 < X < 8) = e^{-.1(5)} - e^{-.1(8)} = e^{-0.5} - e^{-0.8} = .6065 - .4493 = .1572$

56